# Learning From Real Springs

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Abstract

Many springs do not obey Hooke's Law because they are constructed to have an intrinsic tension which must be overcome before normal elongation occurs. This property, well-known to engineers, is universally neglected in elementary physics courses. In fact it can be used to enhance learning and to deepen understanding of potential energy.

## Introduction

Springs play an important role in first year physics courses for a number of reasons. They provide the standard example of a device which can store energy, in the form of elastic potential energy, which can be converted to kinetic energy. Perhaps more important is the fact that the spring force, given by Hooke's Law, can not be treated by the constant acceleration formulae memorized in high school. In fact this tendency to memorize formulae and then apply them when they are not applicable is one of the largest impediments to mastering the fundamentals of physics necessary for a technical education. By utilizing a certain class of springs in the standard laboratory experiment one may go a step farther in the effort to encourage understanding over memorization. Such a spring can be used as a rich source of physics as well as an example of the dangers of utilizing a generally accepted formula without questioning its validity.

## The Standard Perfect Spring

Every elementary physics book contains a section in which a massless spring is considered. The magnitude of the force exerted by the spring exactly satisfies Hooke's Law being proportional to the amount the spring is stretched or compressed, $|x|$. Thus one has $|F| = k|x|$ where $k$ is the spring constant. The corresponding potential energy function is $U(x) = \frac{1}{2}kx^2 + C$. The arbitrary additive constant is normally set equal to zero. In laboratory experiments the spring is usually oriented vertically in order to eliminate friction. Then choosing the origin at the point where the spring has its natural, un-stretched length and vertically down as the $+y$ direction, the spring force is given by $F_y = -ky$. The spring constant can be measured by hanging any mass $m$ from the spring and finding the equilibrium elongation, $y_{eq}$. Then $k = \frac{mg}{y_{eq}}$. Alternatively one may determine the spring constant from the slope of $F_y$, at equilibrium, plotted as a function of the elongation, $y_{eq}$.

The same value for the spring constant would be obtained either way. For an ideal, mass-less spring, the force as a function of $y_{eq}$ would be the line marked *ideal* in figure 1. For the vertical spring the potential energy would now be the sum of the elastic potential energy, $U_e(y) = \frac{1}{2}ky^2 + C_1$, and the gravitational potential energy, $U_g(y) = -mgy + C_2$. Again the constants are normally set equal to zero. The usual experiment has the students verifying the law of conservation of energy by releasing the mass from rest at the un-stretched position, $y = 0$, and measuring the maximum elongation. Since the total energy is zero, given this choice of constants, one has $\frac{1}{2}ky_{max}^2 - mgy_{max} = 0$ or $y_{max} = 2y_{eq}$. Additionally the student can observe the resulting Simple Harmonic Motion and, perhaps by considering the differential equation which results from applying Newton's Law,

$$-ky + mg = m\frac{d^2y}{dt^2}, \tag{1}$$

find the period to be $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$.

The experimental results usually agree reasonably well with the calculated values.

### The Effect of the Spring's Mass

Taking into account the mass of the spring is fairly complicated and usually approximate results, at most, are included in elementary textbooks. Rather than the ideal results for the magnitude of the force of the spring as a function of the equilibrium elongation, a typical graph is illustrated by the line marked *massive* in figure 1. The non-zero intercept with the horizontal axis is attributed to the extension of the spring due to the effect of gravity on the spring itself. An informative treatment of the problem of a massive spring can be found on the Web at the site entitled MathRec[1] There it is shown that taking into account the mass of the spring, $m_s$, changes the static result for $y_{eq}$ from $\frac{mg}{k}$ to $\frac{m+\frac{m_s}{2}}{k}$ and, at least in some approximation, replaces $m$ by $m + \frac{m_s}{3}$ in both the dynamic quantities kinetic energy and period of oscillation. The calculation of the elastic potential energy function is more complicated. This analysis is beyond the scope of an elementary physics course and thus limits the pedagogical benefit of considering the spring's mass for beginning students. Fortunately all the effects of the mass of the spring become negligible when the hanging mass is much larger than the mass of the spring.

### Intrinsic Tension; A Learning Opportunity

Another feature of real springs, well-known to engineers, is initial tension.[2] Depending on how the wire is fed into the coiling machine, one produces a spring which requires a specific load before it begins to obey Hooke's Law. This requirement of an initial load is in fact a desirable property of the spring in order for certain mechanical devices to operate properly. The fact that many metal springs do not behave linearly was brought to the attention of physics teachers by Prior[3] in 1980. His results, shown schematically in figure 2, were used to alert instructors that there was a non-linear region which should be avoided when doing experiments with springs. Fifteen years later Wagner[4] showed that by a suitable heat treatment, this non-linearity could be eliminated and un-treated springs with initial tension could be transformed into ideal springs. His results for the un-treated spring have the same characteristics as Prior's although the spring is about 100 times stiffer. In a note in 1999, Froehle[5] suggested that because of initial tension the relationship between elongation and $F_y$ should be $F_y = -(ky + b)$. The corresponding potential energy $U(y) = \frac{1}{2}ky^2 + by$ contains an extra term which, Froehle warns, can be significant in some applications. Of course this description of the force is only intended to be employed when the hanging body has a mass that is sufficiently large so that elongation occurs. Inappropriately using this force for smaller masses this would imply that the force of the spring, $b$, being larger than $mg$ causes the body to rise!

One can go beyond Froehle's description, afford the students an opportunity to delve deeper into the potential energy function, better fit typical experimental results such as those of Prior and Wagner, and remove the difficulty for small masses. The effect of this initial tension can be modeled in a way that is amenable to calculation by a beginning physics student. In this way a student might see the danger in blindly using memorized formulae. Assume the functional dependence of the force exerted by the vertical spring on the hanging masses is

$$F_y = -k_1 y \qquad for \qquad y \leq S \tag{2}$$

$$F_y = -k_2(y - S) - k_1 S \qquad for \qquad y > S. \tag{3}$$

Here $k_1$ is much larger than $k_2$ and $S$ is normally very small. The critical mass of the body at which the force changes character is given by $m_{crit} = \frac{k_1 S}{g}$. This force, as a function of the equilibrium elongation, is shown, labeled *model* in figure 2. Froehle's description corresponds to $k_1$ going to infinity and $S$ going to zero so that $(k_1 - k_2)S$, henceforth designated by $\beta$, goes to $b$. Although the significance of the initial tension may be small for some springs it is none the less instructive to consider the consequences of this type of force, having one functional form in one region and another in a different region.[6] The potential energy function corresponding to this force is

$$U_1(y) = \frac{1}{2}k_1 y^2 + C_1 \qquad for \qquad y \leq S \tag{4}$$

$$U_2(y) = \frac{1}{2}k_2 y^2 + (k_1 - k_2)Sy + C_2 = \frac{1}{2}k_2 y^2 + \beta y + C_2 \qquad for \qquad y > S. \tag{5}$$

If one were to set the two additive constants equal to zero the potential energy would be discontinuous at $S$. This would correspond to the force being infinite at that point. In order for $U$ to be continuous one must choose $C_2 - C_1 = \frac{1}{2}(k_2 - k_1)S^2 = -\frac{1}{2}\beta S$. Thus, while it is true in most situations that one can set additive constants in the potential energy function equal to zero, this exception demonstrates the importance of understanding why something is possible rather than simply memorizing.

### Consequences of Ignoring Initial Tension

Let us assume that the model accurately describes the behavior of a spring. If the spring constant were then *incorrectly* determined by a measurement of $y_{eq}$ with a single mass, $m_1$, one would have, assuming Hooke's Law, $k_w y_{eq} = m_1 g$ where the subscript $w$

stands for *wrong*. This incorrect spring constant would correspond to the slope of a line from the origin as shown by the dotted line in figure 2. The force of the spring would be given by $F_y = -k_w y$ with the corresponding potential energy $U(y) = \frac{1}{2}k_w y^2$, choosing the additive constant to be zero. Then checking conservation of energy by releasing the body from $y = 0$ with zero velocity and calculating the maximum displacement, $y_{max}$, yields $\frac{1}{2}k_w y^2 - m_1 g y_{max} = 0$ so that $y_{max} = \frac{2m_1 g}{k_w}$ and since $y_{eq} = \frac{m_1 g}{k_w}$, $y_{max} = 2y_{eq}$, i.e. twice the measured value of $y_{eq}$. According to the model, for $m_1 g > k_1 S$, the relationship between the *correct* spring constant and the measured $y_{eq}$ is given by $k_2 = \frac{m_1 g - \beta}{y_{eq}}$. Calculating the maximum extension using the model potential energy and equating the energy with the body at rest at $y = 0$ with the energy at $y_{max}$, one has

$$C_1 = \frac{1}{2}k_2 y_{max}^2 + \beta y_{max} + C_2 - m_1 g y_{max} \tag{6}$$

with the solution

$$y_{max} = \frac{m_1 g - \beta}{k_2} + \sqrt{(\frac{m_1 g - \beta}{k_2})^2 + \frac{\beta S}{k_2}} = y_{eq} + \sqrt{y_{eq}^2 + \frac{\beta S}{k_2}}. \tag{7}$$

Thus the effect of the initial tension in considering conservation of energy is a correction due entirely to the difference in the additive constants in the potential function! If the mass $m_1$ is near $m_{crit}$ the difference between the $y_{max}$ in the model and $2y_{eq}$ can be about 10% for the spring used by Prior and about 20% for the spring used by Wagner.

Using the incorrect spring constant in the description of the simple harmonic motion leads to the angular velocity $\omega = \sqrt{\frac{k_w}{m_1}} = \sqrt{\frac{g}{y_{eq}}}$. According to the model, assuming the motion takes place only for $y > S$, the angular velocity is $\omega = \sqrt{\frac{k_2}{m_1}} = \sqrt{\frac{g}{y_{eq}}(1 - \frac{\beta}{m_1 g})}$. Again the correction is proportional to the difference in the additive constants. In this case, since the correction is not proportional to the small quantity $S$, the difference between the *incorrect* $\omega$ and the model $\omega$ can be quite large. Depending on the mass $m_1$ the difference can be a factor of 2 for the Wagner spring and 1.5 for the spring used by Prior.

If instead of using a single measurement to determine the spring constant, one measured the slope of the $F_y$ vs $y_{eq}$ curve, the correct spring constant, $k_2$, would be obtained, assuming the masses of the bodies were sufficiently large. For the harmonic motion, therefore, the $\omega$ calculated with it would be the same as that given by the model. Then, however, using Hooke's Law to calculate $y_{eq}$ would yield an incorrect result. The difference between the model and the incorrect $y_{eq}$ would be

$$y_{eq}(model) - y_{eq}(calculated) = \frac{m_1 g - \beta}{k_2} - \frac{m_1 g}{k_2} = \frac{-\beta}{k_2}. \tag{8}$$

Since the difference is independent of $m_1$ it can easily be measurable, especially near $m_{crit}$, where it can be over 100%. Additionally, using the potential energy function corresponding to $F_y = -k_2 y$, the maximum extension would incorrectly be given by $\frac{1}{2}k_2 y_{max}^2 - m_1 g y_{max} = 0$ or $y_{max}(calculated) = \frac{2m_1 g}{k_2}$. The difference between this and the value obtained with the model is

$$y_{max}(model) - y_{max}(calculated) = \frac{m_1 g - \beta}{k_2} + \sqrt{(\frac{m_1 g - \beta}{k_2})^2 + \frac{\beta S}{k_2}} - \frac{2m_1 g}{k_2}. \tag{9}$$

This difference can also be of the order of 100% for masses near $m_{crit}$ for either of the springs considered.

## Conclusion

The fact that metal springs have a more complicated behavior than ideal springs, which obey Hooke's Law, can be used to advantage in elementary physics courses. Analyzing data, such as that obtained by Prior or Wagner will demonstrate that "laws" such as Hooke's Law should be questioned and modified if not consistent with experiment. Using a memorized potential energy function without sufficient regard for its origin can lead to erroneous results, especially when the force has different forms in different regions and continuity must be imposed on the overall potential energy function. By suitable choices of the springs and masses employed, significant deviations from the results predicted by memorized formulae can be obtained.

## References

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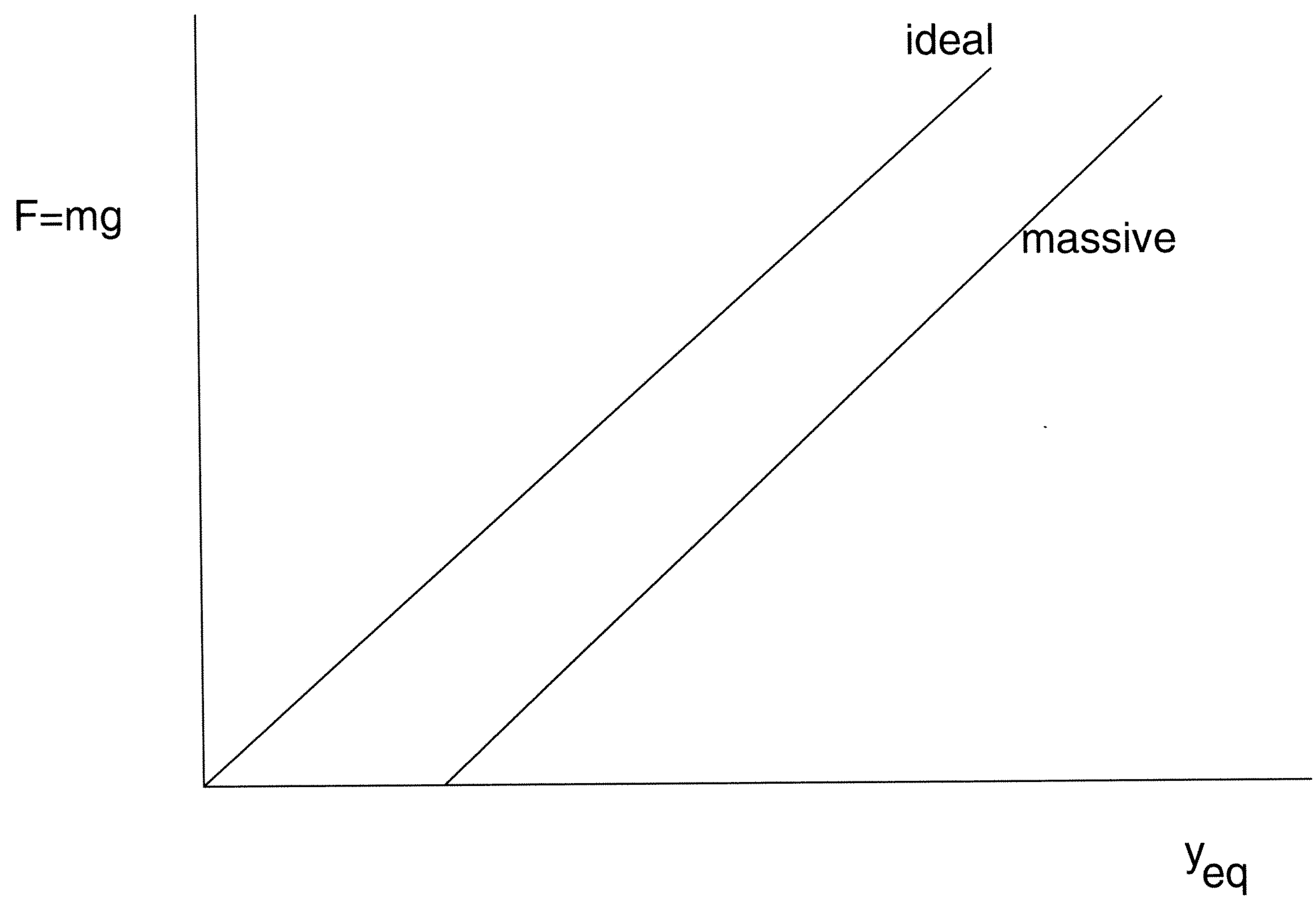
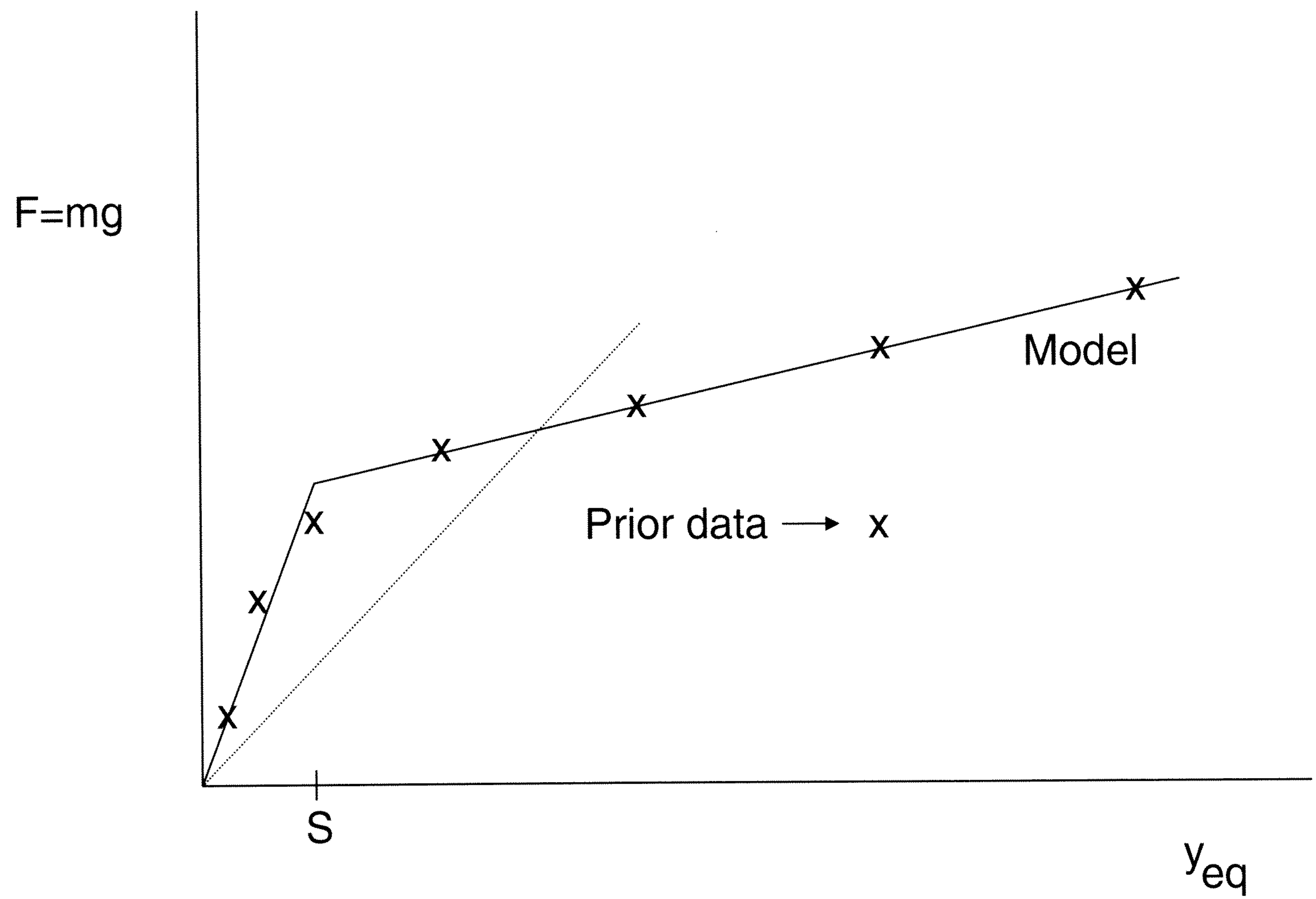
3. R.M. Prior, "A Nonlinear Spring," Phys. Teach **18,** 601 (Nov. 1980)
4. G. Wagner, "Linearizing a Non-Linear Spring," Phys. Teach. **33,** 566 (Dec. 1995)
5. P. Froehle, "Reminder About Hooke's Law and Metal Springs," Phys. Teach. **37**, 368 (Sept. 1999)
6. A falling block with a spring attached to the bottom involves such a force. W. Bassichis, "Don't Panic" (OR Publishing, New York 2005)

**Figure Captions**

Fig.1. For the ideal, massless spring Hooke's Law holds and $|F| = mg = ky_{eq}$, with $m$ the mass of the object hung from the spring. Because of the spring's mass there is an elongation of the spring even when there is no hanging body.

Fig.2. The data of Prior, indicated by x's, showing the nonlinearity of a metal spring. The solid line is the model force which includes the initial tension. The dotted line indicates the incorrect slope that would lead to an incorrect value for the spring constant assuming Hooke's Law and using only one mass.


F=mg
ideal
massive
$y_{eq}$



F=mg
Model
Prior data → x
S
$y_{eq}$